WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER TWO 2017
QUESTIONS OF REVIEW 7: Differential Equations

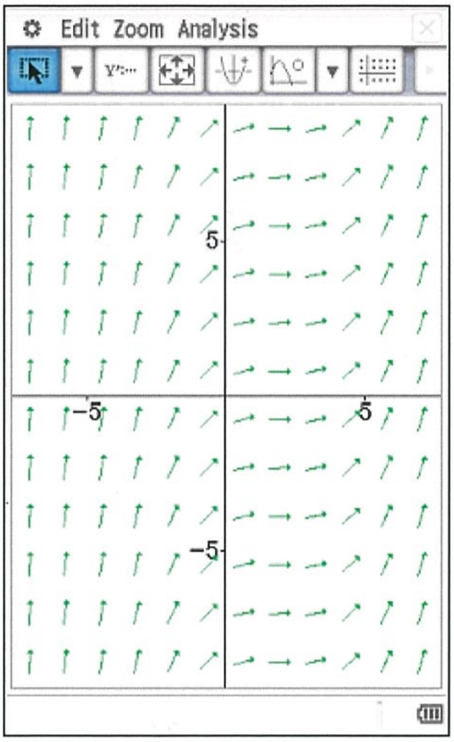
Name: ______________________

Thursday 31st August | Time: 35 minutes | Mark /28

Calculators assumed.

1. [10 marks – 3, 3, 2 and 2] – based on 2016 WACE paper

This slope field represents the family of polynomials $y = f(x) + c$ with $f'(2) = 0$ and $f'(0) = \frac{1}{2}$



a) Explain why $f'(x)$ takes the form $f'(x) = a(x+b)^2$ and evaluate a and b

$f(x)$ is a transformation of x^3

with form $f(x) = A(x+b)^2$

$\therefore f'(x) \sim a(x+b)^2$

$b = -2$

$a = \frac{1}{8}$

b) Use Euler's method, with an increment of $\partial x = 0.2$ to estimate $f(0.6)$, starting from $f(0) = 3$

x	$y = f(x)$	∂x	∂y
0	3	0.2	$\frac{1}{2} \times 0.2 = 0.1$
0.2	3.1	0.2	$\frac{1}{8} \times 1.8^2 \times 0.2 = 0.081$
0.4	3.181	0.2	$\frac{1}{8} \times 1.6^2 \times 0.2 = 0.064$
0.6	3.245	-	-

i.e. $f(0.6) \doteq 3.245$

c) Determine $f(x)$ given $f(0)=3$

$$f'(x) = \frac{1}{8}(x-2)^2$$

$$\therefore f(x) = \frac{1}{24}(x-2)^3 + c$$

$$= \frac{1}{24}(x-2)^3 + \frac{10}{3}$$

d) Compare the actual value of $f(0.6)$ with your estimate in (b) and explain the difference.

$$f(0.6) = \frac{1}{24}(-1.4)^3 + \frac{10}{3} = 3.219$$

Euler's method uses a series of tangents to approximate the curve. In this case, tangents are above a convex curve. $\delta x = 0.2$ is quite large.

2. [18 marks – 2, 4, 2, 1, 1, 1, 1, 3, 1, 1 and 1] – based on 2016 SA Specialist paper

A country's population growth depends on the internal growth rate r and the number of immigrants I, so that $\dfrac{dP}{dt} = rP + I$

(a) Use separation of variables to show that $\displaystyle\int \frac{1}{P+\frac{I}{r}}\,dP = \int r\,dt$

$$\frac{dP}{dt} = rP + I = r\left(P + \frac{I}{r}\right)$$

$$\therefore \frac{dP}{P+\frac{I}{r}} = r\,dt$$

$$\therefore \int \frac{1}{P+\frac{I}{r}}\,dP = \int r\,dt$$

(b) Hence solve the differential equation $\dfrac{dP}{dt} = rP + I$ and show that

$P(t) = P_0 e^{rt} + \dfrac{I}{r}\left(e^{rt} - 1\right)$ where $P(0) = P_0$

d.e. $\Rightarrow \ln\left|P + \frac{I}{r}\right| = rt + c$

$$\therefore P + \frac{I}{r} = e^{rt+c} = C_1 e^{rt}$$

$P(0) = P_0 \Rightarrow P_0 + \frac{I}{r} = C_1$

$\therefore P + \frac{I}{r} = e^{rt}\left(P_0 + \frac{I}{r}\right)$

$\therefore P = e^{rt} P_0 + e^{rt} \cdot \frac{I}{r} - \frac{I}{r}$

$= e^{rt} \cdot P_0 + \frac{I}{r}\left(e^{rt} - 1\right)$

A country in 2017 has $P_0 = 24.5$ million, $I = 0.21$ million and $r = 0.0062$

(c) Estimate the population in 5 years time (the year 2022)

$P(5) = 24.5\, e^{0.0062 \times 5} + \frac{.21}{.0062}\left(e^{0.0062 \times 5} - 1\right)$

$= 26.34$ mill

(d) Estimate the population in 50 years time

$P(50) = 45.71$ mill

(e) Identify one assumption in these estimates

r + I stay the same

(f) How reliable is the model in the longer term?

Not reliable because (1) r + I will probably change
(2) logistic effect are ignored.

A logistic equation for this country's population is derived from $\frac{dP}{dt} = 0.016P\left(\frac{45-P}{45}\right)$ with a 2017 population of $P_0 = 24.5$ million and an increased r value to include immigration.

(g) What is the maximum population?

45 million

(h) Estimate the population in 5 years time (2022) and after 50 years

$P(t=5) = 25.39$ million

$P(t=50) = 32.70$ million

ClassPad: $M = 45$ $P_0 = 24.5$

$k = 0.016/45$

or: $a = 0.016$ $b = \frac{0.016}{45}$

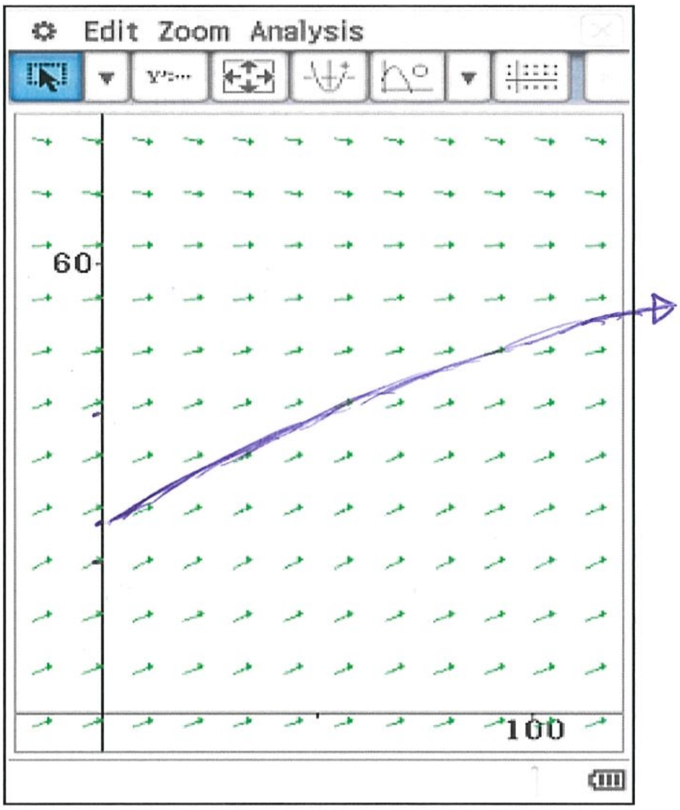
$P_0 = 24.5 \Rightarrow c = 2.975 \times 10^{-4}$

(i) What is the major difference between these logistic results and the earlier model?

logistic has a maximum of 45 mill; other model is unbounded

A combined equation has $\frac{dP}{dt} = 0.0062P\left(\frac{45-P}{45}\right) + 0.21$

(j) Draw the solution curve with $P_0 = 24.5$ million on this slope field.



(k) How do the results from this model compare with the logistic model?

Have higher limit (~65 million) or may actually continue to increase.